Uncertainty Evolutionary Game Based on Sequence of Interval Numbers

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*Abstract:*Most problems in the real world are uncertain. Classical game theory is difficult to solve the decision-making problem with uncertain profits. In this paper, interval numbers are used to express uncertainty. Based on the order relation of interval numbers, an uncertainty evolutionary game model is proposed and the solution method is given. The stability of the model is studied by using the eagle-pigeon game. The research results of this paper enrich the game theory.

1. Introduction

In the classical game theory, the element of profits matrix is an exact constant. The optimal strategy and decision revenue of each player can be calculated by linear program. However, in the real world, the decision profits of most games are uncertain and can only be given by prior probability or estimation of experts, which is difficult for classical game theory in this condition. In recent years, many scholars have carried out researches on uncertain game and made remarkable achievements. Literature [1] proposed an axiomatic definition of similarity measure between the dual hesitant fuzzy set and applied in matrix game. Literature [2] proposed an effective technique for solving constraint matrix games with rough interval payoffs. Literature [3] proposed approaches to solve interval-valued matrix game with the basic idea of splitting an interval game matrix to its lower and upper bounds. Literature [4] using the asymmetrically distributed information to express the cross-evaluated payoffs of players and obtain the solution of the matrix game with cross-evaluated payoffs based on max-min principles. Literature [5] proposed an effective method for solving matrix games with belief structures payoffs based on D-S Theory. Generally, uncertainty game is the combination of uncertain theory and game theory, using different uncertainty description methods to establish game model, and using the game theory to calculate the equilibrium point.

In summary, scholars have made a lot of achievements in the research of uncertainty matrix game [6-8], but there are few attentions to uncertainty evolutionary game[9]. The main difficulty is that the operation between uncertain data is difficult and uncertain data will cause dimension disaster in the iterative solution of evolutionary game. In order to solve these problems, an evolutionary game model and solution method based on interval number sequence relationship is proposed in this paper, the stability of the model is studied by using an uncertain hawk-pigeon game.

2. Interval Numbers and its Sequence

2.1. Interval Numbers

Let $a = [a^L, a^U]$ is a closed interval over a bounded real number field, $a^L, a^U \in R$ and $a^L \leq a^U$, then $a = [a^L, a^U]$ is an interval number. The set of all interval numbers is denoted by I_R, i.e.

$$I_R = \{[a^L, a^U] | a^L \leq a^U, a^L, a^U \in R\} \tag{1}$$

If $a = [a^L, a^U] = \{x | 0 < a^L \leq x \leq a^U\}$, a is called a positive interval number. For any two interval numbers a and b, the basic operation is defined as follows:

$$\begin{cases} a + b = [a^L + b^L, a^U + b^U] \\ a - b = [a^L - b^U, a^U - b^L] \\ ab = [min\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\}, \\ \qquad max\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\}] \\ \frac{a}{b} = [a^L, a^U] \cdot \left[\frac{1}{b^U}, \frac{1}{b^L}\right] \\ \lambda a = \begin{cases} [\lambda a^L, \lambda a^U], \lambda \geq 0 \\ [\lambda a^U, \lambda a^L], \lambda < 0 \end{cases} \end{cases} \tag{2}$$

2.2. Sequence of Interval Numbers

In this paper, the symbol " $\prec$" is used in the comparison of two interval numbers, indicating the relationship between two interval numbers is not greater than, i.e. $\forall a \in I_R, b \in I_R$, if $a \prec b$, it means that b is better than a or at least the same as a. In particular, when $a, b \in R$, $a \prec b$ can be expressed as $a \leq b$, which means the symbol " $\prec$" in the real number field represents the relationship of less than or equal. In literature [10], an interval number sequence is proposed as follows:
When a and b are real numbers, the possibility of $a \prec b$ is defined as:

$$p(a \prec b) = \begin{cases} 1, & a < b \\ \frac{1}{2}, & a = b \\ 0, & a > b \end{cases} \tag{3}$$

When at least one of a or b is not a real number, for interval numbers $a = [a^L, a^U]$, $b = [b^L, b^U]$, Let $l(a) = a^U - a^L$, $l(b) = b^U - b^L$, then the possibility of a $\prec$ b is defined as:

$$p(a \prec b) = 1 - max\{min\left(1, \frac{a^U - b^L}{l(a) + l(b)}\right), 0\} \tag{4}$$

Therefore, for the set composed of all interval numbers I_R, the possibility based on relation " $\prec$" on I_R (abbreviated to no greater than relation in case of no confusion) is defined as follows:

$$\forall a = [a^L, a^U], b = [b^L, b^U] \in I_R : a \precsim b$$
$$\Leftrightarrow p(a \prec b) \geq \frac{1}{2} \tag{5}$$

It can be proved that no greater than relation " $\prec$" is a total ordering relation, and interval number can be sorted by using no greater than relation " $\prec$".

3. Evolutionary Game Based on Interval Numbers

Evolutionary game theory is a theory combining game theory analysis with dynamic evolution process analysis. It not only pays attention to the stable structure of game, but also studies the relationship between the stable structure and evolution process of game system by introducing different dynamic mechanisms. In this chapter, the model and solution method of uncertainty evolutionary game will be proposed based on interval number sequence theory.

3.1.Evolutionary Game

In an evolutionary game, suppose that the strategy space of two players is $\Omega = \{\delta_1, \delta_2, \cdots, \delta_n\}$, where n is the total number of finite strategies, each strategy is regarded as a species, and the proportion of species in the population is $x_1, x_2, \cdots, x_n$, respectively, and satisfies the condition that the population ratio additive sum is 1. e.t. $\sum_{i=1}^{n} x_i = 1$. Then the set $x = \{x_1, x_2, \cdots, x_n\}$ of population proportion can be regarded as the mixed strategy of game players. In the process of evolution, the profits of the competition between species δ_i and species δ_j is defined as a_{ij}, where $1 \le i \le n, 1 \le j \le n$. Then the profits matrix of the population is defined as:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \ddots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \tag{6}$$

The expected profits of species δ_i is defined as:

$$E_i = \sum_{n}^{j} a_{ij} x_j \tag{7}$$

And the actual profits of the population are:

$$E = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i a_{ij} x_j \tag{8}$$

For the evolution mechanism of population, Peter D. and Taylor et al. [11] proposed a dynamic evolution model based on the following dynamic evolution equation:

$$\frac{dx_i}{dt} = x_i (E_i - E) \tag{9}$$

The significance is that over time, when the expected profits of a species is greater than the actual profits, the number (or proportion) of the species tends to increase. Otherwise, it tends to decrease. For game strategy, when the expected profits of the selected pure strategy is greater than the actual profits, the frequency of using the pure strategy will be increased; otherwise, the frequency of using the strategy will be reduced. When $t \to \infty$, if there is a small proportion of ε mutation behavior x in the population, the profits of the mutation population is always smaller than the original population, then the population evolution is stable, and the distribution proportion of species in the population constitutes an evolution stable strategy ESS (10).

$$E(x^*|(1-\varepsilon)x^* + \varepsilon x) > E(x|(1-\varepsilon)x^* + \varepsilon x), x \ne x^* \tag{10}$$

3.2. Uncertainty Evolutionary Game Model based on Interval Numbers

Let $\hat{a}_{ij} \in [A_{ij}^L, A_{ij}^U]$ is the uncertain profits of species i and species j when they meet, then the uncertain profits matrix of the population is defined as:

$$\hat{A} = \begin{pmatrix} \hat{a}_{11} & \cdots & \hat{a}_{1n} \\ \cdots & \ddots & \cdots \\ \hat{a}_{n1} & \cdots & \hat{a}_{nn} \end{pmatrix} \tag{11}$$

The uncertain expected profits of species δ_i is:

$$\hat{E}_i = \sum_n^j \hat{a}_{ij} x_j \tag{12}$$

And the uncertain profits of the population is:

$$\hat{E} = \sum_{i=1}^n \sum_{j=1}^n x_i \hat{a}_{ij} x_j \tag{13}$$

For the evolution mechanism of this uncertainty game, The dynamic evolution equation is proposed in this paper based on eq. (9) as follows:

$$\frac{dx_i}{dt} = x_i [p(\hat{E} < \hat{E}_i) - \frac{1}{2}] \tag{14}$$

The significance is: when the possibility of the uncertain expected profits $\hat{E}$ of the population is not greater than the uncertain expected profits $\hat{E}_i$ of the species δ_i, the number of individuals of the species δ_i has an increasing trend, e.t. the population proportion x_i of the species δ_i increases in the evolution process; otherwise, the population proportion x_i of species δ_i will decrease.

In summary, because of the profits matrix is uncertain, the calculation results of the expected profits of the species and the population are also uncertain. Therefore, in this paper, the population proportion of the species was adjusted by comparing the interval number sequence relationship between the expected profits of single species and the whole population in the evolution process. From the perspective of uncertainty, the species which profits is higher than the expected profits of the whole population are more adaptable to the environment. Therefore, the proportion of this species tends to increase, and the higher the adaptability of this species is, the faster the proportion will rise; otherwise, the proportion will decline.

4. Game Analysis based on the Evolution of Hawk and Pigeon

In order to show the idea of uncertainty evolutionary game more clearly, this paper takes the classic eagle-pigeon game model as an example to study the evolutionary game process under uncertainty description.

4.1. An Uncertainty Game Model of Hawk Pigeon

In an eagle-pigeon game, the profits matrix of the eagle-pigeon game is shown in Table 1, where v represents the total amount of resources owned by two species in a competition. If the eagle meets the eagle, it is considered that the eagle is aggressive and fights with each other, resulting in the loss of part of its profits c, and then the average distribution profits is $(v - c)/2$. If the eagle meets the

pigeon, it is considered that the pigeon are not aggressive which choose defensive strategies, and the eagle will gain all the resources v and the pigeon will gain no profits. If the pigeon meets the pigeon, they will distribute the resources equally. The species in the population is composed of hawk and pigeon, it can be noted that the proportion of hawk is x, and the pigeon proportion is $(1 - x)$.

Table 1: Profits of Eagle-pigeon game.

	hawk	pigeon
hawk	(v − c)/2,(v − c)/2	v,0
pigeon	0,v	v/2,v/2

In the classic game model of hawk and pigeon, the proportion of each species can be calculated through the ESS of the population, by the determined real numbers v and c. Now considering that the resources of two species in a competition are uncertain, noted by interval number $\hat{v}$ and the resources loss caused by two eagles in a competition is also uncertain, noted by interval number $\hat{c}$, then the expected profits of eagle in this game is:

$$E_1 = \frac{\hat{v}-\hat{c}}{2}x + \hat{v}(1 - x) \tag{15}$$

The expected profits of pigeon is:

$$E_2 = \frac{\hat{v}}{2}(1 - x) \tag{16}$$

And the expected profits of the population is:

$$E = xE_1 + (1 - x)E_2 \tag{17}$$

From equation (13), the population evolution equation is:

$$\frac{dx}{dt} = x[p(E \prec E_1) - \frac{1}{2}] \tag{18}$$

In this way, an uncertainty evolutionary game model of the hawk-pigeon expressed by interval number is established. In the next section, this paper will use specific data to study the game model.

4.2. Model Solution and Dynamic Phase Analysis

In the above hawk-pigeon game model, if $\hat{v} = [2,4]$, $\hat{c} = [4,6]$, then the uncertain game profits matrix can be expressed as follows:

Table 2: Profits matrix of uncertain Eagle-pigeon game.

	hawk	pigeon
hawk	[− 4,0],[− 4,0]	[2,4],0
pigeon	0,[2,4]	[1,2],[1,2]

According to equation (15~17), the expected profits of eagle is:
$$E_1 = [- 4,0]x + [2,4](1 - x) = [- 6x + 2, - 4x + 4]$$

The expected profits of pigeon is:
$$E_2 = [1,2](1-x) = [-x+1, -2x+2]$$
And the expected profits of the population is:
$$E = xE_1 + (1-x)E_2 = [-5x^2+1, -2x^2+2]$$
From equation (18), the evolution equation of this example is:
$$F(x) = \frac{dx}{dt} = x\left[p(E < E_1) - \frac{1}{2}\right] = x\left[\frac{1}{2} - max\left\{min\left(1, \frac{-2x^2+6x}{3x^2+2x+3}\right), 0\right\}\right], 0 \le x \le 1$$
The phase diagram of dynamic replication of evolution equation $F(x)$ is shown in "Figure 1".
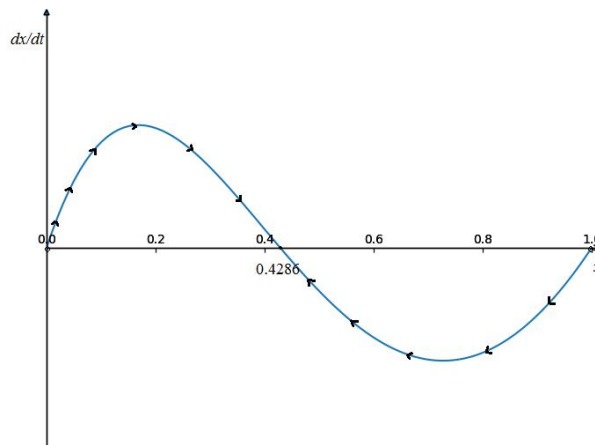

Figure 1: Phase diagram of $F(x)$.

In this game, the fixed points are $x = 0, x = 0.4286$ and $x = 1$, where $x^* = 0.4286$ is stable point, $x = 0$ and $x = 1$ are unstable point. That means, when the evolutionary stability is reached, the population proportion of eagle is about 0.4286, and the population proportion of pigeon is about 0.5714, and the strategy $(0.4286, 0.5714)^*$ is an evolutionary stable strategy (ESS) of the game. At this time, the expected return of eagle is $[-0.5716, 2.2856]$, and the pigeon is $[0.5714, 1.1428]$.

The game model analysis shows that in the evolutionary stability strategy, the profits of hawk strategy and pigeon strategy are also uncertain, but the best profits interval $[0.0815, 1.6326]$ can be achieved by using the evolutionary stable strategy. When one of the game players changes its strategy, the changed strategy profits interval will not greater than the profits interval that the evolutionary stability strategy is adopted.

5. Conclusions

How to make decision in competitive environment is a common and important problem. Game theory provides a basic conceptual framework to model and analyze these problems. Many unknown strategy payments are represented by approximations in really word problems, in order to be more realistic, this paper uses interval numbers to represent uncertain data, the uncertainty evolutionary game of interval number is modeled, and the dynamic evolution equation is proposed based on the sequence of interval number. Finally, the stability of the evolution model is studied by using the phase diagram of the eagle-pigeon game, which enriched the game theory.

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